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The Chern–Simons state for topological invariants

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ABSTRACT

The covariant canonical formalism for the second Chern and Euler topological invariants which depends of a connection valued in the Lie algebra of $SO(3,1)$ is performed. We show that the Chern–Simons state corresponds to an eigenfunction of zero energy for such characteristic classes, in particular, for the Euler class within self-dual (or anti-self-dual) scenario. In addition, to complete our analysis we develop the Hamiltonian analysis for the theories under study, obtaining a best description of the results obtained with the symplectic method.

1. Introduction

Topological field theories are characterized by being devoid of local degrees of freedom. That is, the theories are susceptible only to global degrees of freedom associated with non-trivial topologies of the manifold in which they are defined and topologies of the gauge bundle [1]. We can find in the literature several examples of topological field theories, for example, those called BF theories. As we know, BF theories were introduced as generalizations of three-dimensional Chern–Simons actions or in other cases, can also be consider as a zero coupling limit of Yang–Mills theories [2,3]. On the other side, we find other relevant examples of topological field theories with the called topological invariants such as the second Chern and Euler classes. These topological invariants have been topic of study in many recently works because they are expected to be related to physical observables, as, for instance, in the case of anomalies [4–9].

The importance for studying topological actions has been motived in several contexts of theoretical physics given that has a closed relation with physical theories. One example of this is the well-known MacDowell–Maunsouri formulation of gravity. In this formulation, breaking the $SO(5)$ symmetry of a BF-theory for $SO(5)$ group down to $SO(4)$ we can obtain the Palatini action plus the sum of second Chern and Euler topological invariants. Because these topological classes have trivial local variations that do not contribute classically to the dynamics, we thus obtain essentially general relativity [10].

On the other hand, we can find in the literature the so-called Chern–Simons state which is an other interesting example where a closed relation between a topological field theory and physical theories is present. As we know, the Chern–Simons state corresponds to an exact state of the Schrödinger equation for Yang–Mills theory in four dimension [11]. In addition, we can find in a recent work that the Chern–Simons state describes a topological state with unbroken diffeomorphism invariance in Yang–Mills and general relativity [12]. Furthermore, in the loop quantum gravity context that state is called the Kodama state and has been studied in interesting works by Smolin, arguing that the Kodama state at least for the Sitter spacetime, loop quantum gravity does have a good low energy limit [13].

With these antecedents, the purpose of this Letter is to develop the quantization analysis for the second Chern and the Euler invariants which will be written in terms of a connection valued in the Lie algebra of $SO(3,1)$. In this manner, with this paper we make progress for future works where we will try to quantize the theory reported in [10] using the results found in this article. As part of this work, we will show using the symplectic method that the Chern–Simons state corresponds to an eigenstate of zero energy for these topological invariants, in particular for the Euler class within the self-dual (or anti-self-dual) scenario. Our analysis will be performed in two steps, the first one will be the development of the symplectic formalism, and in the second one we will use the Hamiltonian formalism to obtain relevant physical information. In particular, we reproduce the best description of the results obtained using the symplectic formalism. It is important to mention that recently the covariant canonical analysis for the second Chern class has been reported in [14] with $SU(n)$ as internal symmetry group. However, that analysis is not complete and the identification of the constrains is wrong, thus, the Hamiltonian and the constrains for the theory has not been well identified, therefore, with the results reported in [14] we cannot know important results, as for example the counting of physical degrees of freedom. In this form, this work extends and completes the results reported in [14].

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The Letter is organized as follows. In Section 2, we develop the canonical covariant analysis for the second Chern class which depends only of a connection valued in $SO(3,1)$ symmetry group, and we show that the Chern–Simons state corresponds to an eigenfunction of zero-energy for the theory under study. In Section 3, the Hamiltonian analysis for the second Chern topological invariant is performed. We identify the Hamiltonian, all the constraints for the theory and we carry out the counting of degrees of freedom, showing that the theory under study defines a topological field theory. With these results, we extend and complete the work reported in [14]. In Section 4, we develop the canonical covariant formalism for the Euler class, showing that the Chern–Simons state corresponds to an eigenfunction of zero-energy for the quantum Hamiltonian within the self-self dual (or anti-self-dual) scenario. In the Section 5, we study the Hamiltonian analysis for the Euler invariant, we identify the Hamiltonian, the constraints and we carry out the counting of degrees of freedom for the theory showing that, it defines a topological field theory too. In addition, we obtain the best description of the results obtained in the Section 4. In the Section 6, we give some remarks and conclusions.

2. The symplectic method for the second Chern invariant

As we know, the second Chern topological invariant written in terms of a connection valued in the Lie algebra of $SO(3,1)$ symmetry group is given by [10,11]

$$S_{sc}[\omega]=\alpha\int_M R^{IJ}[\omega]\wedge R_{IJ}[\omega], \tag{1}$$

where $R^{IJ}[\omega]=\frac{1}{2}R^{IJ}{}_{\mu\nu}\,dx^\mu\wedge dx^\nu$ is the curvature of the $SO(3,1)$ 1-form connection $\omega_\nu{}^{IJ}$ with $R^{IJ}{}_{\mu\nu}=\partial_\mu\omega_\nu{}^{IJ}-\partial\nu\omega_\mu{}^{IJ}+\omega_\mu{}^{IK}\omega_{\nu K}{}^J-\omega_\nu{}^{IK}\omega_{\mu K}{}^J$. Here $\mu,\nu=0,1,\ldots,3$ are spacetime indices, x^μ are the coordinates that label the points for the 4-dimensional Minkowski manifold M and $I,J=0,1,\ldots,3$ are internal indices that can be raised and lowered by the internal Lorentzian metric $\eta_{IJ}=(-1,1,1,1)$. Taking the variation of the action S_{sc} we obtain

$$\delta S_{sc}[\omega]=\int d^4x\left(\alpha\varepsilon^{\alpha\beta\mu\nu}D_\beta R_{IJ\mu\nu}\right)\delta\omega_\alpha{}^{IJ}-\int d^4x\,\partial_\alpha\Psi^\alpha_{sc}, \tag{2}$$

where we can identify the equations of motion

$$\varepsilon^{\alpha\beta\mu\nu}D_\beta R_{IJ\mu\nu}=0, \tag{3}$$

and the simplectic potential for the second Chern class [14,15]

$$\Psi^\alpha_{sc}=\alpha\varepsilon^{\alpha\beta\mu\nu}R_{IJ\mu\nu}\delta\omega_\beta{}^{IJ}. \tag{4}$$

Here, $D_\beta R_{IJ\mu\nu}=\partial_\beta R_{IJ\mu\nu}+\omega_\beta{}^K R_{KJ\mu\nu}+\omega_{\beta J}{}^K R_{IK\mu\nu}$ is the covariant partial derivative respect to Lorentz indices and $\varepsilon^{\alpha\beta\mu\nu}$ is the volume 4-form associated with the spacetime metric $\eta_{\mu\nu}$.

Now, for future useful calculations we shall obtain the linearized equations of motion for the Chern invariant. For this purpose, we replace $\omega_\beta{}^{IJ}$ by $\omega_\beta{}^{IJ}\to\omega_\beta{}^{IJ}+\delta\omega_\beta{}^{IJ}$ in (3) and keep only the first-order terms in the $\delta\omega_\beta{}^{IJ}$ fields, obtaining

$$\varepsilon^{\alpha\beta\mu\nu}D_\beta\delta R^{IJ}{}_{\mu\nu}+\varepsilon^{\alpha\beta\mu\nu}\delta\omega_\beta{}^{IK}R_K{}^J{}_{\mu\nu}+\varepsilon^{\alpha\beta\mu\nu}\delta\omega_\beta{}^{JK}R^I{}_{K\mu\nu}=0, \tag{5}$$

which corresponds to the linearized Bianchi's equations.

On the other hand, the kernel integral of the symplectic form is defined by means of the functional exterior derivative of Ψ^α_{sc} on the second Chern class covariant phase space, which is defined as the set of solutions of the classical equations of motion (3) [14,15]. Thus, the symplectic structure is defined by

$$\begin{aligned}\varpi_{sc}&=\int_\Sigma J^\alpha\,d\Sigma_\alpha=\int_\Sigma\delta\Psi^\alpha_{sc}\,d\Sigma_\alpha\\&=\int_\Sigma\delta\left(\varepsilon^{\alpha\beta\mu\nu}R_{IJ\mu\nu}\right)\wedge\delta\omega_\beta{}^{IJ}\,d\Sigma_\alpha,\end{aligned} \tag{6}$$

where Σ is a Cauchy hypersurface. In this manner, with all these results at hand, we can prove that ϖ_{sc} is a closed and Lorentz invariant, as well as a gauge invariant symplectic structure on the phase space of $SO(3,1)$ second Chern class, in a flat spacetime.

To prove that ϖ_{sc} is a closed two-form, we can observe that $\delta^2\omega_\beta{}^{IJ}=0$ because δ is nilpotent [15], thus

$$\begin{aligned}\delta\left(\varepsilon^{\alpha\beta\mu\nu}\delta R^{IJ}{}_{\mu\nu}\right)&=\varepsilon^{\alpha\beta\mu\nu}\delta\left(D_\mu\delta\omega_\nu{}^{IJ}-D_\mu\delta\omega_\nu{}^{IJ}\right)\\&=\varepsilon^{\alpha\beta\mu\nu}\left(\delta\omega_\mu{}^I{}_K\delta\omega_\nu{}^{kJ}+\delta\omega_\mu{}^J{}_K\delta\omega_\nu{}^{IK}\right.\\&\qquad\left.-\delta\omega_\nu{}^I{}_K\delta\omega_\mu{}^{kJ}-\delta\omega_\nu{}^J{}_K\delta\omega_\mu{}^{IK}\right)=0,\end{aligned} \tag{7}$$

where we have used the antisymmetry among 1-forms $\delta\omega_\mu{}^I{}_K$. Therefore

$$\begin{aligned}\delta J^\alpha&=\delta\left(\varepsilon^{\alpha\beta\mu\nu}\delta R_{IJ\mu\nu}\right)\wedge\delta\omega_\beta{}^{IJ}-\left(\varepsilon^{\alpha\beta\mu\nu}\delta R_{IJ\mu\nu}\right)\wedge\delta^2\omega_\beta{}^{IJ}\\&=0\end{aligned} \tag{8}$$

proving that ω_{sc} is closed.

Furthermore, we can see that under a gauge transformation $\omega_\beta{}^{IJ}\to\omega_\beta{}^{IJ}-D_\beta\epsilon^{IJ}$, for some infinitesimal variation we have

$$\delta\omega'_\beta{}^{IJ}=\delta\omega_\beta{}^{IJ}+\delta\omega_\beta{}^I{}_K\epsilon^{KJ}+\delta\omega_\beta{}^J{}_K\epsilon^{IK}, \tag{9}$$

$$\delta R'^{IJ}{}_{\mu\nu}=\delta R^{IJ}{}_{\mu\nu}-\delta R^{KJ}{}_{\mu\nu}\epsilon^I{}_K-\delta R^{IK}{}_{\mu\nu}\epsilon^J{}_K. \tag{10}$$

Using (9) and (10) in ϖ_{sc}, we find that transforms

$$\begin{aligned}\varpi'_{sc}&=\int\delta\left(\varepsilon^{\alpha\beta\mu\nu}R'_{IJ\mu\nu}\right)\wedge\delta\omega'_\beta{}^{IJ}\,d\Sigma_\alpha,\\&=\varpi_{sc}+\int\varepsilon^{\alpha\beta\mu\nu}\left(\delta R_{IJ\mu\nu}\delta\omega_\beta{}^I{}_K\epsilon^{KJ}+\delta R_{IJ\mu\nu}\delta\omega_\beta{}^J{}_K\epsilon^{IK}\right.\\&\qquad\left.-\delta R_{KJ\mu\nu}\delta\omega_\beta{}^{IJ}\epsilon_I{}^K-\delta R_{IK\mu\nu}\delta\omega_\beta{}^{IJ}\epsilon_I{}^K\right)d\Sigma_\alpha+\int O\left(\epsilon^2\right)d\Sigma\\&=\varpi_{sc}+\int O\left(\epsilon^2\right)d\Sigma.\end{aligned} \tag{11}$$

Therefore, ϖ_{sc} is an $SO(3,1)$ singlet. This result allows us to prove that

$$\begin{aligned}\partial_\alpha J^\alpha&=D_\alpha J^\alpha\\&=D_\alpha\left(\varepsilon^{\alpha\beta\mu\nu}\delta R_{IJ\mu\nu}\right)\wedge\delta\omega_\beta{}^{IJ}+\left(\varepsilon^{\alpha\beta\mu\nu}\delta R_{IJ\mu\nu}\right)\wedge D_\alpha\delta\omega_\beta{}^{IJ}\\&=\left(\varepsilon^{\alpha\beta\mu\nu}R_{KJ\mu\nu}\delta\omega_{\beta I}{}^K+\varepsilon^{\alpha\beta\mu\nu}R_{IK\mu\nu}\delta\omega_{\beta J}{}^K\right)\wedge\delta\omega_\alpha{}^{IJ}\\&\quad+\frac{1}{2}\varepsilon^{\alpha\beta\mu\nu}\delta R_{IJ\alpha\beta}\wedge\delta R^{IJ}{}_{\mu\nu}\\&=0,\end{aligned} \tag{12}$$

where we have used the linearized Bianchi equations given in (5) and the antisymmetry of 1-forms $\delta\omega_\beta{}^{IJ}$.

Thus, performing a Lorentz transformation $\Sigma_t\to\Sigma_{t'}$ and $\varpi_{sc}\to\varpi'_{sc}$, this is

$$\varpi'_{sc}=\int_{\Sigma_{t'}}\delta\Psi'^\alpha\,d\Sigma'_\alpha=\int_{\Sigma_t}\delta\Psi^\alpha\,d\Sigma_\alpha=\varpi_{sc}. \tag{13}$$

It follows that ϖ_{sc} given in (6) is Lorentz invariant. In this manner, with these results we have constructed a Lorentz and gauge invariant symplectic structure on the $SO(3,1)$ second Chern phase space and it is possible to formulate the Hamiltonian theory in a manifestly covariant way.

With the symplectic form previously constructed, we can find the energy–momentum tensor by means of the contraction of ϖ_{sc} with a vector field $V = \epsilon^{\mu}\partial_{\mu}$ corresponding to the translation by constant spacetime vector ϵ^{μ} [14]. For this step we need to calculate the next useful contractions

$$V\rfloor\delta\omega_{\beta}{}^{IJ} = \epsilon^{\nu}R^{IJ}{}_{\nu\beta},$$
$$V\rfloor\delta R_{IJ\mu\nu} = \epsilon^{\beta}D_{\beta}R_{IJ\mu\nu}. \tag{14}$$

In this form, the contraction $V\rfloor\varpi_{sc}$ yields

$$V\rfloor\varpi_{sc} = -2\alpha\int\epsilon^{\gamma}\delta\biggl(\varepsilon^{\alpha\beta\mu\nu}R_{IJ\mu\nu}R^{IJ}{}_{\gamma\beta} - \frac{1}{4}\eta^{\alpha}{}_{\gamma}\varepsilon^{\mu\nu\rho\beta}R_{IJ\mu\nu}R^{IJ}{}_{\rho\beta}\biggr)d\Sigma_{\alpha}, \tag{15}$$

where it is possible identify the energy–momentum tensor for the second Chern invariant given by

$$T^{\alpha\gamma} = \alpha\biggl[\varepsilon^{(\alpha\beta\mu\nu}R_{IJ\mu\nu}R^{IJ}{}_{\sigma\beta}\eta^{\sigma\gamma)} - \frac{1}{4}\eta^{\alpha\gamma}\varepsilon^{\mu\nu\rho\beta}R_{IJ\mu\nu}R^{IJ}{}_{\rho\beta}\biggr], \tag{16}$$

which is classically zero, in concordance with the topological invariance of the action. Calculating the T^{00} component of the energy–momentum tensor, we find the density energy for the theory,

$$T^{00} = R^{IJ}{}_{0a}\biggl[\Pi^{a}{}_{IJ} - \frac{\alpha}{2}\eta^{abc}R_{IJabc}\biggr], \tag{17}$$

where $\Pi^{a}{}_{IJ} \equiv \frac{\alpha}{2}\eta^{abc}R_{IJbc}$, $\eta^{abc} = \varepsilon^{0abc}$ and $a, b = 1, 2, 3$. Using the density energy (17) we can find the classical Hamiltonian for the second Chern class [14],

$$H = \int_{\Sigma} R^{IJ}{}_{0a}\biggl[\Pi^{a}{}_{IJ} - \frac{\alpha}{2}\eta^{abc}R_{IJbc}\biggr]d^{3}x. \tag{18}$$

It is remarkable to note that the classical Hamiltonian found by this method strictly differs from the classical Hamiltonian obtained using the Dirac procedure (see Section 3, Eq. (29)), Thus, we can neither identify all the constraints for the theory nor carry out the counting of the physical degrees of freedom. In general, we can see that the form used to identify the constraints reported in [14] for this theory is wrong. Therefore, in this section to avoid mistakes we will not call constraints those which were identified in [14]. To make progress in this point and correct the results reported in [14] we identify the correct constraints for the theory performing the Hamiltonian analysis in the next section.

On the other hand, considering that $d\Sigma_{\mu}$ is time-like vector field which allow us to work with the temporal gauge $\omega_{0}{}^{IJ} = 0$ [14], the symplectic structure (6) takes the form

$$\varpi_{sc} = \int_{\Sigma} 2\delta\Pi^{a}{}_{IJ}\wedge\delta\omega_{a}{}^{IJ}, \tag{19}$$

thus, this expression allows us to identify the next classical–quantum correspondence $2\Pi^{i}{}_{IJ} \to \mathrm{i}\frac{\delta}{\delta\omega_{i}{}^{IJ}}$, this is

$$\bigl(\hat{\omega}_{a}{}^{IJ}(x)\psi\bigr)(\omega) = \omega_{a}{}^{IJ}\psi(\omega),$$
$$\bigl(\hat{\Pi}^{a}{}_{IJ}(x)\psi\bigr)(\omega) = \mathrm{i}\frac{\delta\psi(\omega)}{\delta\omega_{a}{}^{IJ}}, \tag{20}$$

where ψ is an arbitrary function of the connection and represents a quantum state. With this correspondence the classical–quantum Hamiltonian is given by

$$\hat{H} = \int R^{IJ}{}_{0a}\biggl[\mathrm{i}\frac{\delta}{\delta\omega_{a}{}^{IJ}} - \frac{\alpha}{2}\eta^{abc}R_{IJbc}\biggr]. \tag{21}$$

Using this result, the representation of the vacuum for the theory which will correspond to an eigenfunction of zero-energy for the Hamiltonian $\hat{H}$ is determined by

$$\hat{H}\Psi(\omega) = 0, \tag{22}$$

hence, any wave-function satisfying

$$\biggl(\mathrm{i}\frac{\delta}{\delta\omega_{i}{}^{IJ}} - \frac{\alpha}{2}\eta^{ijk}R_{IJjk}\biggr)\Psi(\omega) = 0, \tag{23}$$

satisfies automatically (22). Here $\Psi(\omega)$ solves exactly Eq. (23) and is expressed by

$$\Psi(\omega) = e^{-\mathrm{i}\alpha I(\omega)}, \tag{24}$$

with

$$I(\omega) = \int\frac{1}{2}\omega^{IJ}\wedge d\omega_{IJ} + \frac{1}{3}\omega^{IK}\wedge\omega_{KL}\wedge\omega^{L}{}_{I}, \tag{25}$$

being the Chern–Simons functional. The wave function given in (24) is known in the literature as the Chern–Simons state [11–13] and corresponds to an eigenfunction of zero-energy for the Hamiltonian given in (21) which is the Hamiltonian of the second Chern class found by means of the symplectic method. It is important to remark that in this program we have used the temporal gauge demanding that $d\Sigma_{\mu}$ is time-like vector field. In this manner, we can talk about an eigenstate of zero energy for the Hamiltonian (18). It is important to remark, that the energy–momentum tensor for the theory is classically zero and $\Psi(\omega)$ is an exact zero-energy eigenfunction at quantum level. In this manner, we find a classical–quantum correspondence. However, within the Dirac program the Hamiltonian for the theory is a linear combination of constraints, so we cannot talk about an eigenstate of zero energy for first class Hamiltonians because in these theories we do not have a Schrödinger equation. The action of the Hamiltonian on physical states is annihilation. We will see this in the next section.

3. Hamiltonian analysis for the second Chern invariant

In this section, we will perform the Dirac analysis for the second Chern class and we will find the Hamiltonian and the constraints for the theory. In particular we will reproduce the results developed in the last section.

For our purposes we rewrite the second Chern action in the next form [2,12,16]

$$S[\omega, R] = \int_{M}\alpha R^{IJ}\wedge F_{IJ}[\omega] - \frac{\alpha}{2}\int_{M}R^{IJ}\wedge R_{IJ}, \tag{26}$$

where $F^{IJ}[\omega] = d\omega^{IJ} + \omega^{I}{}_{L}\wedge\omega^{LJ}$ is the curvature of the $SO(3,1)$ connection $\omega_{\mu}{}^{IJ}dx^{\mu}$ and R_{IJ} is a 2-form field that can be fixed by means of the equations of motion [2,16]. The action (26) yields the next equations of motion

$$DR^{IJ} = dR^{IJ} + \omega^{I}{}_{L}\wedge R^{LJ} + \omega^{J}{}_{L}\wedge R^{IL} = 0, \qquad F^{IJ}[\omega] = R^{IJ}, \tag{27}$$

where F_{IJ} satisfies the Bianchi identities $DF^{IJ} = 0$. Using the equation of motion (27) in (26) we eliminate R, obtaining the same action (1) and equations of motion (3).

Performing the Hamiltonian analysis of (26) we find

$$S = \int d^{4}x\biggl[\Pi^{a}{}_{IJ}\dot{\omega}_{a}{}^{IJ} + \omega_{0}{}^{IJ}D_{a}\Pi^{a}{}_{IJ} + R^{IJ}{}_{0a}\biggl(\Pi^{a}{}_{IJ} - \frac{\alpha}{2}\eta^{abc}F_{IJbc}\biggr)\biggr], \tag{28}$$

where $\Pi^{a}{}_{IJ} = \frac{\alpha}{2}\eta^{abc}R_{IJbc}$, $a, b = 1, 2, 3$, $\eta^{123} = 1$ and $D_{a}\Pi^{a}{}_{IJ} = \partial_{a}\Pi^{a}{}_{IJ} + \omega_{a}{}^{IK}\Pi^{a}{}_{KJ} + \omega_{a}{}^{JK}\Pi^{a}{}_{IK}$. The expression given in (28) allows us to identify the Hamiltonian for the theory,

$$H=\int d^3x\left[-\omega_0{}^{IJ}D_a\Pi^a{}_{IJ}-R^{IJ}{}_{0a}\left(\Pi^a{}_{IJ}-\frac{\alpha}{2}\eta^{abc}F_{IJbc}\right)\right], \tag{29}$$

where we can find the next 24 primary constraints

$$\phi_{IJ}=D_a\Pi^a{}_{IJ}\approx 0, \tag{30}$$

$$\phi^a{}_{IJ}=\Pi^a{}_{IJ}-\frac{\alpha}{2}\eta^{abc}F_{IJbc}\approx 0. \tag{31}$$

Consistency requires that their conservation in the time vanishes as well. For this system there are no, secondary constraints. To compute the algebra between the constrains it is convenient to rewrite them

$$\phi_1:=\phi_{IJ}[B]=\int dx^3\,B^{IJ}D_a\Pi^a{}_{IJ}, \tag{32}$$

$$\phi_2:=\phi^a{}_{IJ}[G]=\int dx^3\,G_a{}^{IJ}\left[\Pi^a{}_{IJ}-\frac{\alpha}{2}\eta^{abc}F_{IJbc}\right]. \tag{33}$$

In this manner, the algebra is

$$\begin{aligned}
&\{\phi_1[B_{IJ}],\phi_1[C_{KL}]\}=\int d^3x\left(B_I{}^KC_{KJ}-B_J{}^KC_{KI}\right)\phi^{IJ}\approx 0,\\
&\{\phi_1[B_{IJ}],\phi_2[G_a{}^{IJ}]\}=\int d^3x\left(B^I{}_KG_a{}^{KJ}-B^J{}_KG_a{}^{KI}\right)\phi^a{}_{IJ}\approx 0,\\
&\{\phi_2[Q_b{}^{IJ}],\phi_2[G_a{}^{KL}]\}=0,
\end{aligned} \tag{34}$$

which shows that the constraints are first class. We can see from the constraints that only 18 are independent because of the Bianchi identities $DF^{IJ}=0$, that is $D_a\phi^a{}_{IJ}=\phi_{IJ}$. In this manner, we have 36 canonical variables $(\omega_a{}^{IJ},\Pi^a{}_{IJ})$ and 18 independent first class constraints. We can conclude that the second Chern class is devoid of dynamical degrees of freedom and therefore defines a topological field theory. In addition, the Hamiltonian given in (29) is a linear combination of first class constrains. We can see that all this important physical information was not obtained in [14]. The reason is because with the use of the covariant canonical method we do not have the control to identify the constrains of the theory, therefore we can neither the identify first or second class constrains nor carry out the counting of the degrees of freedom, among other things.

Following to [11,12], from (28) we can identify the corresponding symplectic structure

$$\{\omega_a{}^{IJ}(x),\Pi^b{}_{KL}(y)\}=\frac{1}{2}\delta^b{}_a\left(\delta^I_K\delta^J_L-\delta^I_L\delta^J_K\right)\delta(x-y). \tag{35}$$

Its classical–quantum correspondence is given by

$$[\hat\omega_a{}^{IJ}(x),\hat\Pi^b{}_{KL}(y)]=\frac{\mathrm{i}}{2}\delta^a{}_b\left(\delta^I_K\delta^J_L-\delta^I_L\delta^J_K\right)\delta(x-y). \tag{36}$$

From this expression, we can identify the classical–quantum correspondence $\Pi^a{}_{KL}\to \mathrm{i}\frac{\delta}{\delta\omega_{aKL}}$ that allows us make progress for the quantization. So, we will proceed for the quantization of our theory. As we know, for theories in which the Hamiltonian is a linear combination of constraints as in our case, we cannot construct the Schrödinger equation because the action of the Hamiltonian on physical states is annihilation, in this manner, at quantum level we cannot talk about the eigenstates of energy for the Hamiltonian (29). In Dirac quantization we have that the restriction of our physical state is archived by demanding that

$$\left(D_a\frac{\delta}{\delta\omega_a{}^{IJ}}\right)\psi(\omega)=0, \tag{37}$$

$$\left(\mathrm{i}\frac{\delta}{\delta\omega_a{}^{IJ}}-\frac{\alpha}{2}\eta^{abc}F_{IJbc}\right)\psi(\omega)=0, \tag{38}$$

where we can find the solution given by

$$\psi(\omega)=e^{-\mathrm{i}\alpha I(\omega)}, \tag{39}$$

with

$$I(\omega)=\int\frac{1}{2}\omega^{IJ}\wedge d\omega_{IJ}+\frac{1}{3}\omega^{IK}\wedge\omega_{KL}\wedge\omega^L{}_I, \tag{40}$$

being the Chern–Simons functional. The constrain (37) implies that $\psi(\omega)$ is unchanged under small gauge transformations [2,12] and the constraint (38) is solved exactly by means of (39). Therefore, the Chern–Simons state corresponds to an quantum state for the second Chern class but not an eigenstate of zero energy because we do not have a Schrödinger equation for our theory. It is important to remark that the symplectic and Dirac methods share the same quantum state. Nevertheless, with the symplectic method has been fixed the temporal gauge, and that allows us to talk about an eigenstate of energy for the Hamiltonian. However, we do not have control to identify the constraints for the theory and we cannot carry out the counting of degrees of freedom. On the other hand, with Dirac's method we have relevant information, for example, the identification of the constraints, the counting of degrees of freedom and the information at quantum level given by the constraint equation (37). As conclusion for this section, we can see that the way to identify the constraints as was reported in [14] is wrong. In addition, self-dual (or anti-self-dual) condition has not been involved for this theory, but for the Euler invariant the self-dual condition will be important. This is the topic for the next section.

4. Symplectic method for the Euler class

In this section, as an important part of this work we will develop the canonical covariant analysis for the Euler class which is also absent in the literature.

The Euler class is defined in the next form [2,12,16]

$$S_E[\omega]=\beta\int *R^{IJ}\wedge R_{IJ}, \tag{41}$$

where R_{IJ} is the 2-form curvature defined in above sections, $*R^{IJ}=\frac{1}{2}\epsilon^{IJKL}R_{KL}$ is the dual of R and ϵ^{IJKL} is the volume 4-form associated with the Lorentz metric η_{IJ}, here $\epsilon_{0123}=1$. Since η_{IJ} has signature $(-+++)$, it follows that the square of the duality operator is minus the identity. Thus, the definition of self-duality involves the complex number i. The curvature will be self-dual (anti-self-dual) if and only if $*R^{IJ}=\pm\mathrm{i}R^{IJ}$, this condition will be important when we will try to find the quantum state of our theory.

Calculating the variation of $S_E[\omega]$ we find

$$\delta S_E[\omega]=\int dx^4\left(\beta\epsilon^{\alpha\rho\mu\nu}D_\rho*R_{IJ\mu\nu}\right)\delta\omega_\alpha{}^{IJ}+\int dx^4\,\partial_\alpha\Psi^\alpha_E, \tag{42}$$

where the equations of motion are given by

$$\epsilon^{\alpha\rho\mu\nu}D_\rho*R_{IJ\mu\nu}=0, \tag{43}$$

and

$$\Psi^\alpha_E=\beta\epsilon^{\alpha\rho\mu\nu}*R_{IJ\mu\nu}\delta\omega_\rho{}^{IJ}, \tag{44}$$

is identified as the symplectic potential for the theory. We can see that after applying the "$*$" product in the equations of motion (43) we obtain the same that second Chern invariant (5). In this manner, Euler and second Chern class share the same equations of motion and therefore the same covariant phase space defined by the set of solutions to the equations of motion (43). However, the symplectic structures of both theories will be quite different. We can see related results in [17] within the study of BF theories and [18] within string theory context.

The linearized equations for the Euler class which can be useful, for example, to prove the closeness of ϖ_E, are given by

$$\varepsilon^{\alpha\rho\mu\nu} D_\beta \delta * R^{IJ}{}_{\mu\nu} + \varepsilon^{\alpha\rho\mu\nu} \delta\omega_\rho{}^{IK} * R_K{}^J{}_{\mu\nu} + \varepsilon^{\alpha\rho\mu\nu} \delta\omega_\rho{}^{JK} * R^I{}_{K\mu\nu} = 0. \tag{45}$$

On the other hand, such as was developed in previous sections, with the symplectic potential given in (44) we can construct the symplectic form for the Euler class by means of its exterior derivative, this is

$$\varpi_E = \int_\Sigma \delta\Psi^\alpha \, d\Sigma_\alpha = \int_\Sigma J^\alpha \, d\Sigma_\alpha = \int_\Sigma \delta\left(\epsilon^{\alpha\rho\mu\nu} * R_{IJ\mu\nu}\right) \wedge \delta\omega_\rho{}^{IJ} \, d\Sigma_\alpha. \tag{46}$$

It is remarkable to note that in spite of Euler and second Chern–Simons class sharing the same equations of motion, the symplectic structures (6) and (46) are quite different, therefore, we expect different results in regard to the second Chern invariant when we try to quantize the theory. We will show this fact in the next lines.

Following the same steps given in above section with the second Chern class. It is straightforward to prove that ϖ_E is closed, gauge and Lorentz invariant as well. Thus, we can use these results to do the quantization of the theory under study.

Using the important contractions

$$V \rfloor \delta\omega_\beta{}^{IJ} = \epsilon^\nu R^{IJ}{}_{\nu\mu}, \qquad V \rfloor \delta * R_{IJ\mu\nu} = \epsilon^\beta D_\beta * R_{IJ\mu\nu}, \tag{47}$$

where the vector field $V = \epsilon^\mu \partial_\mu$ corresponds to the translation by constant spacetime vector ϵ^μ, the contraction of V with the geometric form yields

$$V \rfloor \varpi = -2\beta \int \epsilon_\gamma \delta\left(\varepsilon^{(\alpha\beta\mu\nu} * R_{IJ\mu\nu} R^{IJ}{}_{\sigma\beta}\eta^{\sigma\gamma)} - \frac{1}{4}\eta^{\alpha\gamma}\varepsilon^{\mu\nu\rho\beta} * R_{IJ\mu\nu} R^{IJ}{}_{\rho\beta}\right) d\Sigma_\alpha. \tag{48}$$

From the last equation, we can identify the energy–momentum tensor for the Euler class as

$$T^{\alpha\gamma} = \beta\left[\varepsilon^{(\alpha\beta\mu\nu} * R_{IJ\mu\nu} R^{IJ}{}_{\sigma\beta}\eta^{\sigma\gamma)} - \frac{1}{4}\eta^{\alpha\gamma}\varepsilon^{\mu\nu\rho\beta} * R_{IJ\mu\nu} R^{IJ}{}_{\rho\beta}\right], \tag{49}$$

which is classically zero, in concordance with the topological invariance of the action. Just as in the second Chern class we need to calculate the T^{00} component of the energy–momentum tensor to obtain the energy density for the theory

$$T^{00} = R^{IJ}{}_{0a}\left[\Pi^a{}_{IJ} - \frac{\beta}{2}\eta^{abc} * R_{IJbc}\right], \tag{50}$$

where $\Pi^a{}_{IJ} \equiv \frac{\beta}{2}\eta^{abc} * R_{IJbc}$ and $a, b = 1, 2, 3$. Thus, the classical Hamiltonian for the Euler class is given by

$$H = \int_\Sigma R^{IJ}{}_{0i}\left[\Pi^a{}_{IJ} - \frac{\beta}{2}\eta^{abc} * R_{IJbc}\right]. \tag{51}$$

Just as in the Chern invariant case, we can see that the expression (51) strictly differs from that obtained by means of the Dirac's method (see the next section, Eq. (64)). In this manner, if we ignore the Hamiltonian method we cannot have a full study of the theory, therefore, we can neither know the physical degrees of freedom nor the relevant symmetries. To complete the analysis of the theory under study we will perform the Hamiltonian method for the Euler class in the next section.

Nonetheless, consider that $d\Sigma_\mu$ is time-like vector field, which allows us to work with the temporal gauge $\omega_0{}^{IJ} = 0$ [14], the symplectic structure ϖ_E given in (46) takes the form

$$\varpi_E = \int_\Sigma 2\delta\Pi^a{}_{IJ} \wedge \delta\omega_a{}^{IJ}, \tag{52}$$

thus, we can identify the next classical–quantum correspondence $2\Pi^a{}_{IJ} \to \mathrm{i}\frac{\delta}{\delta\omega_a{}^{IJ}}$, this is

$$\left(\hat\omega_a{}^{IJ}(x)\psi\right)(\omega) = \omega_a{}^{IJ}\psi(\omega), \qquad \left(2\beta\hat\Pi^a{}_{IJ}(x)\psi\right)(\omega) = \mathrm{i}\frac{\delta\psi(\omega)}{\delta\omega_a{}^{IJ}}, \tag{53}$$

where $\psi(\omega)$ is an arbitrary function of the connection and represents a quantum state. Considering this fact, the classical–quantum Hamiltonian is given by

$$\hat H = \int_\Sigma R^{IJ}{}_{0i}\left[\mathrm{i}\frac{\delta}{\delta\omega_a{}^{IJ}} - \frac{\beta}{2}\eta^{abc} * R_{IJjk}\right]. \tag{54}$$

With these results, we can see that the vacuum of the theory which correspond to the eigenfunction of zero-energy for the Hamiltonian $\hat H$ will be expressed by

$$\hat H\Psi(\omega) = 0, \tag{55}$$

hence, any wave-function satisfying

$$\left(\mathrm{i}\frac{\delta}{\delta\omega_a{}^{IJ}} - \frac{\beta}{2}\eta^{abc} * R_{IJbc}\right)\Psi(\omega) = 0, \tag{56}$$

satisfies automatically (55). Here $\Psi(\omega)$ solves exactly equation (56) and is given by

$$\Psi(\omega) = e^{\pm\beta I(\omega)}, \tag{57}$$

with

$$I(\omega) = \int \frac{1}{2}\omega^{IJ} \wedge d\omega_{IJ} + \frac{1}{3}\omega^{IK} \wedge \omega_{KL} \wedge \omega^L{}_I, \tag{58}$$

being the Chern–Simons functional. From (56) we can see that the Chern–Simons state defined in (57) will correspond to an eigenfunction of zero-energy for the Euler class if $*R^{IJ} = \pm \mathrm{i} R^{IJ}$. This is the self-dual (or anti-self-dual) condition. We can observe that in this work the second Chern invariant and the Euler class share the same equations of motion, but not the same Chern–Simons state because the wave-function (57) does not has the presence of the complex number i. This is a good example where two theories share to the same equations of motion, but his respectively quantum theories are different. In this manner, there is a similar relation as the presented between Yang–Mills theory and the second Chern invariant [12]. However, in this work we have a relation between complete topological field theories, because we have showed in the Section 3 that the second Chern is a topological field theory and we will show in Section 5 that Euler class defines a topological field theory as well. In this sense, this Letter is quiet different to that reported in [12] because the relation found in that work was between Yang–Mills theory which is not topological (this theory has $2n$ degrees of freedom) and the second Chern invariant which is a topological one.

5. Hamiltonian analysis for the Euler class

In this part, we shall develop the Hamiltonian analysis for the Euler topological invariant and we will identify the Hamiltonian and the constraints for theory. For our purposes we start rewriting the Euler action given in (41) in the next form [2,12,16]

$$S[\omega, R] = \beta \int_M *R^{IJ} \wedge F_{IJ}[\omega] - \frac{\beta}{2} \int_M *R^{IJ} \wedge R_{IJ}, \tag{59}$$

where $F^{IJ}[\omega] = d\omega^{IJ} + \omega^I{}_L \wedge \omega^{LJ}$ is curvature of the connection $\omega_\mu{}^{IJ} dx^\mu$, $*R^{IJ} = \frac{1}{2}\epsilon^{IJKL} R_{KL}$. Just as the second Chern class, R^{IJ} is a 2-form field that can be fixed means the equations of motion. The action given in (59) implies the equations of motion

$$D * R^{IJ} = 0, \qquad *F^{IJ}[\omega] = *R^{IJ}, \tag{60}$$

where F_{IJ} satisfies the Bianchi identities $DF^{IJ} = 0$. It is straightforward to prove that using Eqs. (60) in (59) we can recover (41) and (43) [2,16].

Performing the Hamiltonian analysis of (59) we find

$$S = \int d^4x \biggl[\Pi^a{}_{IJ} \dot{\omega}_a{}^{IJ} + D_a \Pi^a{}_{IJ} \omega_0{}^{IJ} + R^{IJ}{}_{0a} \biggl(\Pi^a{}_{IJ} - \frac{\alpha}{2} \eta^{abc} * F_{IJbc} \biggr) \biggr], \tag{61}$$

where $\Pi^a{}_{IJ} = \frac{\beta}{2} \eta^{abc} * R_{IJbc}$, $a, b = 1, 2, 3$, $\eta^{123} = 1$ and $D_a \Pi^a{}_{IJ} = \partial_a \Pi^a{}_{IJ} + \omega_a{}^{IK} \Pi^a{}_{KJ} + \omega_a{}^{JK} \Pi^a{}_{IK}$. From (61) we can identify the Hamiltonian for the theory as

$$H = -D_a \Pi^a{}_{IJ} \omega_0{}^{IJ} - R^{IJ}{}_{0a} \biggl(\Pi^a{}_{IJ} - \frac{\alpha}{2} \eta^{abc} * F_{IJbc} \biggr), \tag{62}$$

and the 24 primary constraints are

$$\varphi_{IJ} := D_a \Pi^a{}_{IJ} \approx 0, \tag{63}$$

$$\varphi^a{}_{IJ} := \Pi^a{}_{IJ} - \frac{\beta}{2} \eta^{abc} * F_{IJbc} \approx 0. \tag{64}$$

Consistency requires their conservation in the time vanishes as well. For this system there are no, secondary constraints. To compute the algebra between the constraints it is convenient to rewrite them as

$$\varphi_1 := \varphi_{IJ}[B] = \int dx^3\, B^{IJ} D_a \Pi^a{}_{IJ}, \tag{65}$$

$$\varphi_2 := \varphi^a{}_{IJ}[G] = \int dx^3\, G_a{}^{IJ} \biggl[\Pi^a{}_{IJ} - \frac{\beta}{2} \eta^{abc} * F_{IJbc} \biggr]. \tag{66}$$

In this manner the algebra is

$$\begin{aligned} &\{\varphi_1[B_{IJ}], \varphi_1[C_{KL}]\} = \int d^3x \left(B_I{}^K C_{KJ} - B_J{}^K C_{KI}\right) \varphi^{IJ} \approx 0, \\ &\{\varphi_1[B_{IJ}], \varphi_2[G_a{}^{IJ}]\} = \int d^3x \left(B^I{}_K G_a{}^{KJ} - B^J{}_K G_a{}^{KI}\right) \varphi^a{}_{IJ} \approx 0, \\ &\{\varphi_2[Q_b{}^{IJ}], \varphi_2[G_{aKL}]\} = 0, \end{aligned} \tag{67}$$

which shows that the constraints are first class. We can see that of the last constraints only 18 are independent because of the Bianchi identities $DF^{IJ} = 0$, this is $D_a \phi^a{}_{IJ} = \phi_{IJ}$. In this manner, we have 36 canonical variables $(\omega_a{}^{IJ}, \Pi^a{}_{IJ})$ and 18 independent first class constraints. We can conclude that the Euler class is devoid of dynamical degrees of freedom and therefore defines a topological field theory.

From (61), we can identify the symplectic structure

$$\{\omega_a{}^{IJ}(x), \Pi^b{}_{KL}(y)\} = \frac{1}{2} \delta^b{}_a \left(\delta^I_K \delta^J_L - \delta^I_L \delta^J_K\right) \delta(x - y). \tag{68}$$

The classical–quantum correspondence is

$$\left[\hat{\omega}_a{}^{IJ}(x), \hat{\Pi}^b{}_{KL}(y)\right] = \frac{\mathrm{i}}{2} \delta^a{}_b \left(\delta^I_K \delta^J_L - \delta^I_L \delta^J_K\right) \delta(x - y). \tag{69}$$

This allow us to use the correspondence $\Pi^a{}_{KL} \to \mathrm{i} \frac{\delta}{\delta \omega_a{}^{KL}}$. Now, just as in Section 3 for the second Chern invariant, we can see that at quantum level for the Euler class we cannot talk about an eigenstate of zero energy for the Hamiltonian (62) because is a linear combination of constraints. Thus, the action of the Hamiltonian on physical states is annihilation. To procedure with the quantization using the Dirac program, we can see that our state space is achieved by demanding that

$$\biggl(D_a \frac{\delta}{\delta \omega_a{}^{IJ}} \biggr) \psi(\omega) = 0, \tag{70}$$

$$\biggl(\mathrm{i} \frac{\delta}{\delta \omega_a{}^{IJ}} - \frac{\beta}{2} \eta^{abc} * F_{IJbc} \biggr) \psi(\omega) = 0, \tag{71}$$

where we find the solution $\psi(\omega)$ defined by

$$\psi(\omega) = e^{\pm \beta I(\omega)}, \tag{72}$$

with

$$I(\omega) = \int \frac{1}{2} \omega^{IJ} \wedge d\omega_{IJ} + \frac{1}{3} \omega^{IK} \wedge \omega_{KL} \wedge \omega^L{}_I, \tag{73}$$

being the Chern–Simons functional. Again, the constraint (70) implies that ψ is unchanged under small gauge transformations [2,12] and the expression (71) is solved exactly by means of $\psi(\omega)$ given in (72) if the self-dual (or anti-self-dual) $*F^{IJ} = \pm \mathrm{i} F^{IJ}$ condition hold. In (72), the $(+)$ or $(-)$ signs correspond to for the self-dual or anti-self-dual conditions, respectively. Therefore, in Dirac's method the Chern–Simons state corresponds to a quantum state for the Euler class within self-dual (anti-self-dual) scenario but not to an eigenstate of zero energy.

6. Conclusions and prospects

In this paper, the symplectic and the Hamiltonian methods for the second Chern and Euler classes have been performed. As part of the results found using the symplectic method, we could construct a gauge and Lorentz invariant symplectic form for the theories under study. This allowed us to find, for example, that by working with the temporal gauge the Chern–Simons state corresponds to a zero energy eigenfunction for the quantum Hamiltonian of such theories. In particular, in this work we found that the theories share the same equations of motion but not the same quantum state. On the other hand, in contrast with the symplectic method the Hamiltonian analysis of such topological classes, which is absent in the literature, allowed us to find the Hamiltonian and the correct constraints. With this information we could carry out the counting of the physical degrees of freedom of both theories, concluding that they define a topological field theory. In addition, within the Dirac quantization program we could identify the quantum state for our theories, concluding that the Chern–Simons state corresponds to a quantum state but not a zero energy eigenfunction for the quantum Hamiltonian. The reason is because we do not have a Schrödinger equation for theories with Hamiltonians being a linear combination of first class constraints. Another important point to remark is that in this Letter we corrected the results reported in [14] where the analysis was incomplete.

To finish this paper, it is important to mention that the Chern–Simons state (or Kodama state) presents problems as: It is non-renormalizable and is not invariant under CPT transformations [19], thus, we cannot argue that the Chern–Simons state corresponds to the ground state for the theory. However, with the results reported here we can perform the symplectic and Hamiltonian analysis for the theory reported in [10] which describes general relativity, hoping to contribute in forthcoming works to understand and answer open questions as: Why such quantum state exist? Is the Chern–Simons state normalizable using real variables as in this work? – expecting to find a closed quantum relation between general relativity and topological field theories.

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